STRESS WAVES IN AN ELASTIC PLATE

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The solution of problems on the fracture of solids under the action of impulsive dynamic loadings encounters great difficulties. Even before the onset of fracture the process becomes very complicated because different parts of the body are governed by different "stress-strain" relation, and the position of the boundaries between these parts changes with time. When cracks appear, which lead to new time-dependent boundary conditions, one cannot even speak of any kind of rigorous solution of the problem.

This paper is concerned with the study of dynamic stresses in a medium obeying Hooke's law. In materials whose elastic limit and ultimate strength are close to one another (under dynamic loading) such studies can identify points from which the formation of cracks in the first stage of fracture will originate. If, however, there exists a noticeable interval between the elastic limit and the ultimate strength, where the material is subjected to some inelastic deformations, then these studies can identify instants of time and parts of the body where dangerous stresses can be expected with the greatest probability.

The rigorous solution also becomes useful in the evaluation of some approximate theories found in the literature. It shows, for instance, that the acoustic theory [1,2] is inadequate and the quasi-static theory [3] is limited in its applicability.

Dynamic problems of the theory of elasticity in the case of a half-space, a single-layer or a multi-layered medium with plane-parallel separation boundaries were studied in many papers. A major portion of these papers is devoted to the development and also the the qualitative and quantitative study of formulas for the displacement-vector components. The transition to stresses somewhat complicates the solutions. This complication, however, is unimportant in analogous studies.

1. Let us introduce cylindrical coordinates r, θ, z, Assume that a normal unit force, which varies in time as

$$\varepsilon(t) = 0 \quad \text{при } t < 0, \qquad \varepsilon(t) = 1 \quad \text{при } t > 0 \tag{1.1}$$

is applied at the plate boundary $z = 0$ at the point $r = 0$, $z = 0$, and the boundary $z = h$ is stress-free. At the beginning this will cause the formation of direct longitudinal p-waves and transverse s-waves, which characterize the wave-field in a half-space $z > 0$.

After impinging upon the boundary $z = h$, the p-wave produces a reflected longitudinal pp- and a transverse ps-wave. The s-wave, on the other hand, forms correspondingly a longitudinal sp- and a transverse ss-wave. At the next instant there will take place a reflection from the boudary $z = 0$ and then again from the boundary $z = h$, etc.

For brevity and convenience of comparing the obtained results with the results of previous papers [1,2,4] devoted to the explanation of spalling phenomena, let us study for the time being dynamic processes along the axis of symmetry and limit ourselves to the analysis only of the direct waves p and s, and also the reflected pp-, ps-, sp- and ss-waves.

It should be noted that the above-mentioned waves were studied by Zvolinskii. However, he evaluated asymptotically the stress field only in the neighborhood of the wave-front. Such estimates can be justified to any extent only for points that lie at distances from the origin considerably greater than the wave-length. Quantitative studies of waves of such kind in a layer lying on a liquid half-space are contained in [4]. However, in this paper as well as in [1,2], which were devoted to acoustic problems, a number of inaccuracies were admitted, which caused incorrect calculations of the wave-field intensities.

Formulas for the displacement-vector components, corresponding to different waves, can be written immediately, for instance, according to [5], in the form of double integrals. After carrying out the differentation under the integral sign (the permissibility of which was repeatedly discussed in papers by Petrashen') and using the Lamé formulas, one could easily obtain expressions for the stresses. Finally, after representing these expressions in terms of single integrals [6] along a contour l which runs along the imaginary axis in the right half-plane of the complex variable ζ, and letting $r = 0$, we find the principal stress σ_z in the following form:

For incident waves $(p + s)$

$$\sigma_z = \frac{1}{2\pi^2 bi} \frac{\partial}{\partial t} \int_l \left\{ \frac{g^2}{2\zeta^2 R\,[bt\zeta - z\alpha]} - \frac{2\alpha\beta}{\zeta^2 R\,[bt\,\zeta - z\beta]} \right\} d\zeta \qquad (1.2)$$

For reflected waves $(pp + ps + sp + ss)$

$$\sigma_z = \frac{-1}{2\pi^2 bi} \frac{\partial}{\partial t} \int_l \left\{ \frac{g^2 T}{2\zeta^2 R^2\,[bt\,\zeta - (2h - z)\,\alpha]} - \frac{4g^2\alpha\beta}{\zeta^2 R^2\,[bt\,\zeta - h\alpha - (h - z)\,\beta]} - \right.$$

$$\left. - \frac{4g^2\,\alpha\beta}{\zeta^2 R^2\,[bt\,\zeta - h\beta - (h - z)\,\alpha]} + \frac{2\alpha\beta T}{\zeta^2 R^2\,[bt\,\zeta - (2h - z)\,\beta]} \right\} d\zeta \qquad (1.3)$$

where

$$\gamma = \frac{b}{a} = \sqrt{\frac{1 - 2\sigma}{2\,(1 - \sigma)}}, \qquad \alpha = \sqrt{1 + \gamma^2\zeta^2}, \qquad \beta = \sqrt{1 + \zeta^2}$$
$$g = 2 + \zeta^2, \qquad R = g^2 - 4\alpha\beta, \qquad T = g^2 + 4\alpha\beta \qquad (1.4)$$

and where b represents the velocity of propagation of transverse waves, a is the velocity of propagation of longitudinal waves and σ is Poisson's ratio. As was shown in [5], the branches of the radicals a and β should be fixed by the conditions

$$\arg\alpha = \arg\beta = 0 \quad \text{при } \zeta > 0 \qquad (1.5)$$

2. Let us introduce the following notation

$$\tau = bt\,/\,h, \qquad k = z\,/\,h \qquad (2.1)$$
$$\varphi_1 = \tau\zeta - k\alpha, \qquad \varphi_2 = \tau\zeta - k\beta \qquad (2.2)$$
$$\varphi_3 = \tau\zeta - (2 - k)\,\alpha, \qquad \varphi_4 = \tau\zeta - \alpha - (1 - k)\,\beta \qquad (2.3)$$
$$\varphi_5 = \tau\zeta - \beta - (1 - k)\,\alpha, \qquad \varphi_6 = \tau\zeta - (2 - k)\,\beta \qquad (2.4)$$
$$\varkappa_1 = \varepsilon\,(\tau - k\gamma), \qquad \varkappa_2 = \varepsilon\,(\tau - k) \qquad (2.5)$$
$$\varkappa_3 = \varepsilon\,[\tau - (2 - k)\,\gamma], \qquad \varkappa_4 = \varepsilon\,[\tau - (\gamma + 1 - k)] \qquad (2.6)$$
$$\varkappa_5 = \varepsilon\,[\tau - 1 - (1 - k)\,\gamma], \quad \varkappa_6 = \varepsilon\,[\tau - (2 - k)] \qquad (2.7)$$

Let us place the factor h^{-1} in (1.2) and (1.3) before the integral sign and carry out the integration by residues at points ζ_ν of the right-hand half-plane ζ, which are real roots of the corresponding equations

$$\varphi_\nu(\zeta) = 0 \qquad (\nu = 1, 2, 3, 4, 5, 6) \qquad (2.8)$$

where $\phi_\nu(\zeta)$ are determined from (2.2) to (2.4).

If we assume that

$$\varphi_1'(\zeta_1) = \frac{k}{\zeta_1\sqrt{1 + \gamma^2\zeta_1^2}}, \qquad \varphi_2'(\zeta_2) = \frac{k}{\zeta_2\sqrt{1 + \zeta_2^2}} \qquad (2.9)$$

we obtain for the direct [incident] $(p + s)$ waves

$$\sigma_z = -\frac{1}{\pi h^2 k}\frac{\partial}{\partial\tau}\left\{\varkappa_1 \frac{\alpha g^2}{2\zeta R}\bigg|_{\zeta=\zeta_1} - \varkappa_2 \frac{2\alpha\beta^2}{\zeta R}\bigg|_{\zeta=\zeta_2}\right\} \tag{2.10}$$

where

$$\zeta_1 = \frac{k}{\sqrt{\tau^2 - k^2\gamma^2}}, \qquad \zeta_2 = \frac{k}{\sqrt{\tau^2 - k^2}} \tag{2.11}$$

By using the equations

$$\varphi_3'(\zeta_3) = \frac{2-k}{\zeta_3\sqrt{1+\gamma^2\zeta_3^2}}, \qquad \varphi_4'(\zeta_4) = \frac{1}{\zeta_4\sqrt{1+\gamma^2\zeta_4^2}} + \frac{1-k}{\zeta_4\sqrt{1+\zeta_4^2}} \tag{2.12}$$

$$\varphi_5'(\zeta_5) = \frac{1-k}{\zeta_5\sqrt{1+\gamma^2\zeta_5^2}} + \frac{1}{\zeta_5\sqrt{1+\zeta_5^2}}, \qquad \varphi_6'(\zeta_6) = \frac{2-k}{\zeta_6\sqrt{1+\zeta_6^2}} \tag{2.13}$$

we obtain for the reflected $(pp + ps + sp + ss)$ waves

$$\sigma_z = \frac{1}{\pi h^2}\frac{\partial}{\partial\tau}\left\{\varkappa_3 \frac{1}{2-k}\frac{\alpha g^2 T}{2\zeta R^2}\bigg|_{\zeta=\zeta_3} - \varkappa_4 \frac{4\alpha^2\beta^2 g^2}{[(1-k)\alpha+\beta]\zeta R^2}\bigg|_{\zeta=\zeta_4} - \right.$$
$$\left. - \varkappa_5 \frac{4\alpha^2\beta^2 g^2}{[\alpha + (1-k)\beta]\zeta R^2}\bigg|_{\zeta=\zeta_5} + \varkappa_6 \frac{1}{2-k}\frac{2\alpha\beta^2 T}{\zeta R^2}\bigg|_{\zeta=\zeta_6}\right\} \tag{2.14}$$

Here

$$\zeta_3 = \frac{2-k}{\sqrt{\tau^2 - (2-k)^2\gamma^2}} \tag{2.15}$$

$$\zeta_4 = \left(\frac{-k[\gamma^2-(1-k)^2](2-k) + (k^2-2k+2)\tau^2 + 2(1-k)\tau\sqrt{\tau^2 + k(1-\gamma^2)(2-k)}}{\tau^4 - 2[\gamma^2 + (1-k)^2]\tau^2 + [\gamma^2 - (1-k)^2]^2}\right)^{\frac{1}{2}} \tag{2.16}$$

$$\varphi_5 = \left(\frac{-k[1-\gamma^2(1-k)^2](2-k) + (k^2-2k+2)\tau^2 + 2(1-k)\tau\sqrt{\tau^2 - k(1-\gamma^2)(2-k)}}{\tau^4 - 2[1+\gamma^2(1-k)^2]\tau^2 + [1-\gamma^2(1-k)^2]^2}\right)^{\frac{1}{2}} \tag{2.17}$$

$$\zeta_6 = \frac{2-k}{\sqrt{\tau^2 - (2-k)^2}} \tag{2.18}$$

The computation of the expressions in the braces of formulas (2.10) and (2.14) can be accomplished as well as the computation of the displacements [7].

Note that the values of ζ_4 and ζ_5 can practically most be conveniently found not from Formulas (2.16) and (2.17) but directly from the equations $\phi_4 = 0$, $\phi_5 = 0$ by tabulating the functions $r(\zeta_4)$, $r(\zeta_5)$.

Similarly, we obtain equations for the other two principal stresses σ_r and σ_θ along the axis of symmetry:

In the case of direct waves

$$\sigma_r = \sigma_\theta = -\frac{1}{\pi h^2 k}\frac{\partial}{\partial\tau}\left\{\varkappa_1 \frac{\alpha g[(1-2\gamma^2)\zeta^2-1]}{2\zeta R}\bigg|_{\zeta=\zeta_1} - \varkappa_2\frac{\alpha\beta^2}{\zeta R}\bigg|_{\zeta=\zeta_2}\right\} \qquad (2.19)$$

In the case of reflected waves

$$\sigma_r = \sigma_\theta = \frac{1}{\pi h^2}\frac{\partial}{\partial\tau}\left\{\varkappa_3 \frac{1}{2-k}\frac{g\alpha[(1-2\gamma^2)\zeta^2-1]T}{2\zeta R^2}\bigg|_{\zeta=\zeta_3} + \varkappa_4 \frac{2g^2\alpha^2\beta^2}{[(1-k)\alpha+\beta]\zeta R^2}\bigg|_{\zeta=\zeta_4} - \right.$$

$$\left. - \varkappa_5 \frac{4g\alpha^2\beta^2[(1-2\gamma^2)\zeta^2-1]}{[\alpha+(1-k)\beta]\zeta R^2}\bigg|_{\zeta=\zeta_5} - \varkappa_6\frac{1}{2-k}\frac{\alpha\beta^2 T}{\zeta R^2}\bigg|_{\zeta=\zeta_6}\right\} \qquad (2.20)$$

3. Now we shall study the stresses (2.10) and (2.14) and compare them with the corresponding stresses which result from the acoustic theory. An exact expression for the latter can be easily obtained from Formulas (2.10) and (2.14) if all but the first terms in these formulas are discarded, a limiting process $b \to 0$ is performed, and it is assumed that the derivative of κ_1 and κ_3 (in the generalized sense) represents the Dirac delta-function. This expression has the following form:

For direct waves

$$\sigma_z = -\frac{1}{2\pi}\left\{\frac{1}{h^2k^2}\varepsilon\left(t-\frac{hk}{a}\right) + \frac{1}{ahk}\delta\left(t-\frac{hk}{a}\right)\right\} \qquad (3.1)$$

For reflected waves

$$\sigma_z = \frac{1}{2\pi}\left\{\frac{1}{h^2(2-k)^2}\varepsilon\left(t-\frac{h(2-k)}{a}\right) + \frac{1}{ah(2-k)}\delta\left(t-\frac{h(2-k)}{a}\right)\right\} \qquad (3.2)$$

where according to the notation (2.1)

$$hk = z, \qquad h(2-k) = 2h - z$$

Note that in these formulas the inaccuracies tolerated in [1, 2] have been corrected.

For the convenience of comparison we carry out the differentation in (2.10) and (2.14). There we shall assume that

$$\frac{\partial\zeta_\nu}{\partial\tau} = -\frac{\zeta_\nu}{\varphi'(\zeta_\nu)} \qquad (3.3)$$

Then, taking into account (2.9), (2.12) and (2.13), and also assuming that ζ_ν at the front of the νth wave approaches infinity, one can rewrite Formulas (2.10) and (2.14) in the form

$$\sigma_z = -\frac{1}{\pi h^2 k}\left\{\frac{h}{2a}\delta\left(t-\frac{hk}{a}\right)-\varkappa_1\left[\frac{\zeta^2\alpha}{k}\left(\frac{\alpha g^2}{2\zeta R}\right)'\right]_{\zeta=\zeta_1}+\varkappa_2\left[\frac{\zeta^2\beta}{k}\left(\frac{2\alpha\beta^2}{\zeta R}\right)'\right]_{\zeta=\zeta_2}\right\} \tag{3.4}$$

$$\begin{aligned}\sigma_z = \frac{1}{\pi h^2}\Bigg\{&\frac{h}{2a(2-k)}\delta\left(t-\frac{h(2-k)}{a}\right)-\varkappa_3\left[\frac{\zeta^2\alpha}{(2-k)^2}\left(\frac{\alpha g^2 T}{2\zeta R^2}\right)'\right]_{\zeta=\zeta_3}+\\ &+\varkappa_4\left[\frac{\zeta^2\alpha\beta}{(1-k)\alpha+\beta}\left(\frac{4\alpha^2\beta^2g^2}{[(1-k)\alpha+\beta]\zeta R^2}\right)\right]'_{\zeta=\zeta_4}+\\ &+\varkappa_5\left[\frac{\zeta^2\alpha\beta}{(1-k)\beta+\alpha}\left(\frac{4\alpha^2\beta^2g^2}{[(1-k\beta)+\alpha]\zeta R^2}\right)'\right]_{\zeta=\zeta_5}-\varkappa_6\left[\frac{\zeta^2\beta}{(2-k)^2}\left(\frac{2\alpha\beta^2T}{\zeta R^2}\right)_{\zeta=\zeta_6}\right]\Bigg\}\end{aligned} \tag{3.5}$$

It can easily be verified that only terms containing the Dirac delta-function coincide in the compared formulas at the front of the *p*- and *pp*-waves. After the passage of the above-mentioned fronts, Equations (3.1) and (3.2) maintain a constant value at the same time as (3.4) and (3.5) turn out to be variable functions of time, where (3.4) has a discontinuity of the type of a finite jump at the instant of the passage of the transverse *s*-wave, and (3.5) at the instant of the passage of the converted *ps*- and *sp*-waves and the transverse *s*-wave. The magnitude of these finite jumps at the front of the transverse and converted waves and also the magnitudes of the stresses immediately behind the fronts of the longitudinal waves can be easily determined from (3.4) and (3.5) by a limiting process $\zeta_\nu \to \infty$.

Let us study in some greater detail the direct [incident] waves. After passing to the limit in Formula (3.4) and taking into account detailed calculations of the displacement field, which were performed in [7], one can describe schematically the change of the stress σ_z along the axis of symmetry in the following manner. After passing the front of the *p*-wave (with a singularity of the type of the Dirac delta-function) the stress becomes immediately $(1 + 8\gamma^3)$ times greater than the acoustic approximation (3.1). After that the stress grows monotonically with time, say, parabolically, and up to the instant of the introduction of the transverse *s*-wave it reaches a value of a magnitude several times greater than the original one. At the time of a passage of the transverse *s*-wave-front the stress σ_z drops by a step to 8γ of the values of (3.1) and then varies smoothly. Quickly it reaches a value near the static stress, which is equal to three times the value of (3.1).

The acoustic approximation can be regarded as a particular case of the dynamic theory of elasticity, and thus its application to solids should already on the basis of the performed calculations be considered inadmissible.

In the following, however, the stresses from the acoustic approximation will still be used in order to draw, by means of simple examples, a better qualitative picture of a number of phenomena which take place in solids but are described by more complicated formulas. The calculations by means of these formulas do not present great difficulties, and they can be carried out to an arbitrary degree of accuracy if necessary.

4. In order to utilize the study of the wave-fields for practical applications it is still necessary to perform a transition from the solution $\sigma_z(t)$, which was obtained considering (1.1), to the solution $\sigma_z{}^*(t)$, which corresponds to an "arbitrary" physically reasonable loading $P(t)$ applied at the instant $t = 0$. Such a transition can be performed by means of the well-known formula

$$\sigma_z{}^*(t) = \int_0^t \sigma_z(u) P'(t-u)du \tag{4.1}$$

It should be noted that, depending on the relations between the duration of the action T, the instant t_0 of the introduction of the longitudinal wave and the time t, the integral (4.1) is being evaluated differently. In the case $t > (t_0 + T)$ the lower limit in (4.1) has to be replaced by $(t - T)$, since for $u > t$ and $u < (t - T)$ the equation $P'(t - u) = 0$ holds. In the case $t < (t_0 + T)$, however, the value of the integral (4.1) is given in two terms. The first term corresponds to the integration along the segment $t_0 + 0,\ t$, since $\sigma_z(u) = 0$ at $u < t_0$. Here, the symbol $t_0 + 0$ denotes that the point $t = t_0$ at which there exists a singularity, is excluded from the interval of integration. The second term, on the other hand, is equal to the integral of the expression which contains the Dirac delta-function $\delta(u - t_0)$ as a factor. If $\sigma_z(u)$ has the initial value $\sigma_r{}^\circ(u)$ equal to zero for $u < t_0$, then the second term will be equal to $\sigma_z{}^\circ(t_0)\, P'(t - t_0)$. From this it follows that the solution (4.1) should contain the same singularities as the derivative of the source function $P(t)$. In order to obtain physically sensible continuous values of $\sigma_z{}^*(t)$ (which do not even have a discontinuity of the type of a finite jump) it is necessary to require continuity not only of the function $P(t)$ but also of its derivative $P'(t)$. Such a limitation on the function $P(t)$, which is usually determined from experiment, is entirely permissible. Even with as concentrated loadings as one may desire, the pressure itself and the rate of change of the pressure can be assumed to vary smoothly during some intervals of time that are sometimes unnoticeable to the experimenter. The assumptions of strictly concentrated and instantaneous (jump-like) changes of the loadings $P(t)$ (and sometimes also their derivatives) are allowable as an intermediate stage in mathematical abstractions in a theoretical study of practical problems.

In the solution of complicated problems of the dynamic theory of elasticity it is inconvenient to use Formula (4.1) since $\sigma_z(u)$ contains, for instance in (3.4) and (3.5), derivatives of quite complicated expressions. The initial values $\sigma_z^{\circ}(u)$, however, which appear in the solutions as derivatives (2.10) and (2.14) differ from the corresponding displacement solutions of the problems by very simple multipliers. It is easier to handle them not only because they can be more easily calculated, but also because one can use the results of analyses performed in terms of displacements.

Let us assume for the sake of simplicity that $P'(0) = P'(T) = 0$ and perform the integration in (4.1) by parts. As a result of this we obtain a formula which is convenient for numerical integration and which differs from (4.1) by the fact that it contains in the integrand the initial value $\sigma_z^{\circ}(u)$ instead of $\sigma_z(u)$ and the second derivative $P''(t-u)$ instead of $P'(t-u)$.

Thus, in the case $t < (t_0 + T)$, the stress can be written in the following two forms:

$$\sigma_z^*(t) = \begin{cases} \int\limits_{t_0+0}^{t} \sigma_z^{\circ}(u)\, P'(t-u)\, du + \sigma^{\circ}_z(t_0)\, P'(t-t_0) \\ \int\limits_{t_0+0}^{t} \sigma_z^{\circ}(u)\, P''(t-u)\, du \end{cases} \tag{4.2}$$

In the case $t > (t_0 + T)$, on the other hand, the lower limit of integration in (4.2) is automatically replaced by $(t - T)$ and the term $\sigma_z^{\circ}(t_0)\, P'(t \text{ y } \bar{t}_0)$ goes to zero, since the function P and its derivatives are equal to zero if $(t-u) > T$ and $(t - t_0) > T$. In the solution of the acoustic problem, however, we find for direct waves from (3.2) and (4.1)

$$\sigma_z^*(t) = -\frac{1}{2\pi}\left\{\frac{1}{z^2} P\left(t - \frac{z}{a}\right) + \frac{1}{az} P'\left(t - \frac{z}{a}\right)\right\} \tag{4.3}$$

The formula for reflected waves differs from (4.3) by the sign and the pressure of the parameter $(2h - z)$ instead of z.

5. Let us take as an example a continuous loading $P(t)$ of a duration $(c + d)$ in the form of a broken line

$$P(t) = \begin{cases} 0 & (t \leqslant 0) \\ \frac{t}{c} & (0 \leqslant t \leqslant c) \\ \left(1 + \frac{c}{d}\right) - \frac{t}{d} & (c \leqslant t \leqslant c+d) \\ 0 & (t \geqslant c+d) \end{cases} \tag{5.1}$$

If c is much smaller than d this line can be assumed to correspond basically to $P(t)$ load-curves measured in impacts and explosions. Of course, the function (5.1) cannot be attributed to a physically sensible loading since it has finite discontinuities of derivatives at $t = 0$, $t = c$ and $t = (c + d)$. This will lead to the result that the corresponding stress field will have discontinuities of the type of a finite jump. However, if we agree to smooth out slightly the corners of line (5.1) we do not admit a large error if we retain the previous stress-field but only replace the above-mentioned jumps by continuous, rapidly varying, monotonic curves similar to the jumps.

The form of the loading (5.1) is favorable also for the following reasons. First, by means of limiting process $c \to 0$ one can obtain a solution for a loading with an instantaneous increase of the load, as assumed in [1, 2, 4], and compare our solution with solutions obtained in these references. Second, for such a loading the integral (4.1) is used in finite form, since in the integrand $P'(t - u)$ takes on only constant values, and in $\sigma_z(u)$ the initial value $\sigma_z^\circ(u)$, which is always known, is, as a rule, simpler than $\sigma_z(u)$ itself.

Indeed, from (5.1) we obtain

$$P'(t) = \begin{cases} 0 & (t < 0) \\ \frac{1}{c}\varepsilon(t) & (0 < t < c) \\ -\frac{1}{d}\varepsilon(t-c) & (c < t < c+d) \\ 0 & (t > c+d) \end{cases} \tag{5.2}$$

where ϵ denotes the function (1.1). After replacing t in (5.2) by $(t - u)$ and then substituting (5.2) into (4.1), where by $\sigma_z(u)$ one understands $\partial_z^\circ/\partial u$, and assuming that in the lower limit for $u < t_0$, where t_0 is the instant of the introduction of the front, the initial value is zero, we find the basic solution formulas

$$\sigma z\ (t) = \begin{cases} 0 & (t<t_0) \\ \frac{1}{c}\sigma_z{}^\circ(t) & (t_0<t<t_0+c) \\ \frac{1}{c}\sigma_z{}^\circ(t) - \left(\frac{1}{c}+\frac{1}{d}\right)\sigma_z{}^\circ(t-c) & (t_0+c<t<t_0+c+d\ , \\ \frac{1}{c}\sigma_z{}^\circ(t) - \left(\frac{1}{c}+\frac{1}{d}\right)\sigma_z{}^\circ(t-c) + \frac{1}{d}\sigma_z{}^\circ(t-c-d) & (t_0+c+d<t) \end{cases} \tag{5.3}$$

By means of these general formulas, or, more simply, by the substitution of (5.1) and (5.2) into (4.3) we find the following for the direct waves in the acoustic approximations:

$$\sigma_z{}^*(t) = \begin{cases} -\frac{1}{2\pi}\left[\frac{1}{z^2}\frac{1}{c}\left(t-\frac{z}{a}\right)+\frac{1}{za}\frac{1}{c}\right]. & \left(\frac{z}{a}<t<\frac{z}{a}+c\right) \\ -\frac{1}{2\pi}\left\{\frac{1}{z^2}\left[1+\frac{c}{d}-\frac{1}{d}\left(t-\frac{z}{a}\right)\right]-\frac{1}{za}\frac{1}{d}\right\} & \left(\frac{z}{a}+c<t<\frac{z}{a}+c+d\right) \end{cases} \tag{5.4}$$

By combining (5.4) with a similar formula for reflected waves one can calculate the total stress-field at any point as a function of time t.

One can see from (5.4) that at the head of the wave-front there exist only compressive stresses which reach their maximum values at $t = z/a + c$. After a reflection from the boundary $z = h$, a similar tensile stress is formed. For small c these stresses are very large, and for $c = 0$ they contain a singularity of the type of a Dirac delta-function.

Note that in [1, 2, 4] the part of the wave-field that goes over to the Dirac delta-function for $c \to 0$ is not taken into account. The maximum tensile stresses computed from Formula (5.4) appear at the tail of the front. However, depending on the choice of $P(t)$ they may be absent or may appear somewhere in the middle of the front.

It would be easy to compute the stresses by means of Formulas (5.3) and (2.10) for solid media along the axis of symmetry. In this calculation a term would appear which would contain the derivative $P'(t - z/a)$, just as in (4.3). The remaining part of the wave-field would agree in sign, but it would not coincide in its form (depending on z) with the first term of (4.3). It would be at least 2 to 3 times greater in intensity than this term. The stresses in a solid have only some qualitative resemblance to the stresses in the acoustic approximation.

There is no reason to doubt that the tensile stresses in the reflected wave play a fundamental role in the formation of rear spalling. The formation of face spalling, however, cannot be studied without a knowledge of the exact form of the function $P(t)$ and without considering surface and transverse waves away from the z-axis, where they are more intensive than along the z-axis according to [7].

6. In the analysis of problems of fracture caused by dynamic loadings one attempts sometimes [3] to bring in a so-called quasi-static solution which represents the product of the static solution and the loading function $P(t)$. Such an approach to the problem is, of course, completely justified if $P(t)$ changes smoothly during a sufficiently long period of time T. Then, at every moment $t = t^*$ the stress field in the medium will be approximately equal to that static state which would prevail under a constant load $P(t^*)$. In the case of sharply varying $P(t)$, however, the possibility of using the quasi-static solution can in each given case be justified or discarded only after computing the stresses by means of a rigorous dynamic theory of elasticity for the given loadings, boundary conditions and distances from the source of vibrations.

At first, let us study the particular case of a medium in which the shear modulus μ is close to zero, and we shall analyze only the stress-field of the direct waves. For $\mu \to 0$ it coincides, as was already mentioned, with the acoustic solution (4.3).

The acoustic solution contains the conditional "quasi-static" part of the wave-field clearly as the first term of Formula (4.3). It propagates with the speed a and varies with distance as z^{-2}. From

$$\frac{1}{a} P'\left(t - \frac{z}{a}\right) \Big/ \frac{1}{z} P\left(t - \frac{z}{a}\right)$$

one can judge by how much the second term of (4.3) is smaller than the first.

Of course, this relation is as small as is desired for arbitrary t for sufficiently small z and continuous $P'(t)/P(t)$.

In the case of the solid medium (for $\mu \neq 0$) the stress formulas contain the part of the wave-field which is equal to the second term in (4.3) in an explicit form. The quasi-static solution clearly does not appear explicitly in the formulas of the stress-field in the half-space. For an exact comparison with the quasi-static solution one should compute this field for given $P(t)$.

As a rough estimate, however, on the basis of the analysis of stresses performed above and of the displacement graphs given in [7], one can assume it to be equal to a doubled field along the axis of symmetry in an infinite medium [8].

Note that analyses of a free layer reported in this paper are also easily applied to multi-layered media. In the particular case of such media, if one layer borders on a liquid or solid half-space, all formulas derived above remain correct if, in the terms in braces of (2.14) and (2.19), one introduces, according to [5], additional factors which contain the coefficients of reflection from the interface boundary.

Note, also, that if the functions $\phi_\nu(\zeta) - 1/nh$ are introduced instead of the functions of the form (2.2) to (2.4), these formulas will correspond to a bell-like distribution, depending on n, of the loading along the boundary [6].

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